The Expression of any Differential Coefficient of a Function of a Function of any number of Variables by aid of the corresponding Differential Coefficients of any n Powers of the Function, where n is the Order of the Differential Coefficient.

By J. C. Fields.

Let $x, y, z \dots$ be any number of independent variables, u any function of these variables, and Δ any function of the degree n of the symbols $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \dots$, n being a positive integer. Evidently

$$\left(\frac{d}{dx}\right)^n u^m = g(m) \cdot u^m; \left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \dots u^m = h(m) \cdot u^m,$$

where $g(m)$, $h(m)$, considered with regard to m alone, are polynomials in this quantity of the degrees n and $r+s+\dots$ respectively, whence plainly $\Delta u^m = f(m) u^m$ where $f(m)$ is a polynomial of the degree n in m. We have therefore, by the ordinary theory of partial fractions,

$$1. \quad \Delta u^m = f(m) u^m = \theta(m) \cdot \frac{f(m)}{\theta(m)} u^m = \theta(m) u^m \sum_0^p {}_r \frac{f(\alpha_r)}{\theta'(\alpha_r)} \cdot \frac{1}{m - \alpha_r}$$

$$= \theta(m) \sum_0^p {}_r \frac{u^{m-\alpha_r}}{m - \alpha_r} \cdot \frac{f(\alpha_r) u^{\alpha_r}}{\theta'(\alpha_r)} = \theta(m) \sum_0^p {}_r \frac{u^{m-\alpha_r}}{m - \alpha_r} \cdot \frac{\Delta u^{\alpha_r}}{\theta'(\alpha_r)},$$

where $\theta(m) = (m - \alpha_0)(m - \alpha_1) \dots (m - \alpha_p)$, the α's being any constant quantities all distinct from one another and greater than n in number. In particular, if we put $p = n$, $\alpha_r = r$ for all values of r, $\Delta \equiv \left(\frac{d}{dx}\right)^\rho \left(\frac{d}{dy}\right)^\sigma \left(\frac{d}{dz}\right)^\tau \dots$ where $\rho + \sigma + \tau + \dots = n$, and designate by $B_{m, \rho, \sigma \dots}$ the coefficient of $x^\rho y^\sigma z^\tau \dots$ in

the development of u^m, we have on dividing equation (1) through by $\rho!\,\sigma!\,\tau!\,\ldots$, putting $x=0$, $y=0$, $z=0\,\ldots$, and noticing that $\theta'(r)=(-1)^{n-r}r!\,(n-r)!$

$$2.\qquad B_{m,\rho,\sigma\ldots} = \theta(m)\sum_0^n{}_r \frac{u_0^{m-r}}{m-r}\cdot\frac{B_{r,\rho,\sigma\ldots}}{\theta'(r)}$$

$$= m(m-1)\ldots(m-n)\sum_1^n{}_r \frac{u_0^{m-r}}{m-r}\cdot\frac{(-1)^{n-r}B_{r,\rho,\sigma,\ldots}}{r!\,(n-r)!},$$

where u_0 expresses the value of u when $x=0$, $y=0$, $z=0\,\ldots$

In particular, if u be a function of one variable (x) only, and $\Delta\equiv\left(\frac{d}{dx}\right)^n$,

$$B_{m,n} = m(m-1)\ldots(m-n)\sum_1^n{}_r \frac{u_0^{m-r}}{m-r}\cdot\frac{(-1)^{n-r}B_{r,n}}{r!\,(n-r)!},$$

a formula originally due to a suggestion of Eisenstein, and of which a proof has been given by Sylvester.*

We might notice more generally that $\Delta u^m v = F(m)\,u^m$ where u and v are any functions of the variables and $F(m)$ is a polynomial of the n^{th} degree in m, whence, in the same way as (1), we derive

$$3.\qquad \Delta u^m v = \theta(m)\sum_0^p{}_r \frac{u^{m-\alpha_r}}{m-\alpha_r}\cdot\frac{\Delta u^{\alpha_r}v}{\theta'(\alpha_r)}.$$

Hence we see that formula (2) holds more generally where $B_{m,\rho,\sigma\ldots}$ is the coefficient of $x^\rho y^\sigma\ldots$ in $u^m v$.

If we have $\phi(u)=\sum a_m u^m$ (the a's being constant coefficients), and apply (1), we get

$$4.\qquad \Delta\phi(u) = \sum_0^p{}_r \frac{u^{-\alpha_r}\Delta u^{\alpha_r}}{\theta'(\alpha_r)}\left(\sum{}_m \frac{\theta(m)}{m-\alpha_r}a_m u^m\right) = \sum_0^p{}_r \frac{u^{-\alpha_r}\Delta u^{\alpha_r}}{\theta'(\alpha_r)}\cdot\frac{\theta\left(u\frac{d}{du}\right)}{\frac{d}{du}-\alpha_r}\phi(u).$$

If, in (4), we put $p=n$, and $\alpha_r=r$ for all values of r, then

$$\theta\left(u\frac{d}{du}\right) = u\frac{d}{du}\left(u\frac{d}{du}-1\right)\left(u\frac{d}{du}-2\right)\ldots\left(u\frac{d}{du}-n\right) = u^{n+1}\left(\frac{d}{du}\right)^{n+1}$$

* Quarterly Journ. of Math., Vol. I, p. 199. Also Bertrand, Cal. Diff., p. 131.

and we have

5. $$\Delta\phi(u) = \sum_0^n {}_r \frac{(-1)^{n-r} u^{-r} \Delta u^r}{r!\,(n-r)!} \cdot \frac{u^{n+1}\left(\frac{d}{du}\right)^{n+1}}{u\frac{d}{du} - r} \phi(u)$$

$$= u^{n+1} \sum_0^n {}_r \frac{(-1)^{n-r} u^{-r} \Delta u^r}{r!\,(n-r)!} \left(\frac{d}{du}\right)^{n+1} \cdot u^r \int u^{-(r+1)} \phi(u)\, du$$

$$= u^{n+1} \sum_0^n {}_r \sum_0^r {}_s \frac{(-1)^{n-r} u^{-r} \Delta u^r}{(n-r)!\,(r-s)!} \left(\frac{d}{du}\right)^{n-s} \frac{\phi(u)}{u^{s+1}},$$

or if in (5) we operate first with $u^{n+1}\left(\frac{d}{du}\right)^{n+1}$ and afterwards with $\left(u\frac{d}{du} - r\right)^{-1}$, we get

6. $$\Delta\phi(u) = \sum_0^n {}_r \frac{(-1)^{n-r} \Delta u^r}{r!\,(n-r)!} \int u^{n-r} \phi^{(n+1)}(u)\, du,$$

where, in performing the integration by successive partial integrations, we put zero for the arbitrary constant, or what is the same thing, the integral is the sum of a number of terms each one of which contains as a factor $\phi(u)$ or some differential coefficient of $\phi(u)$, as is readily seen from (5), where the whole operation upon $\phi(u)$ is direct and not inverse; thus from (6) we obtain

7. $$\Delta\phi(u) = \sum_0^n {}_\rho \sum_0^\rho {}_r \frac{(-1)^{\rho-r} u^{\rho-r} \Delta u^r}{r!\,(\rho-r)!} \phi^\rho(u) = \sum_0^n {}_r \sum_r^n {}_\rho \frac{(-1)^{\rho-r} u^{\rho-r} \phi^\rho(u)}{r!\,(\rho-r)!} . \Delta u^r.$$

For example, we find

$$\Delta e^u = e^u \sum_0^n {}_r \sum_r^n {}_\rho \frac{(-1)^{\rho-r} u^{\rho-r} \Delta u^r}{r!\,(\rho-r)!},$$

$$\Delta \log u = -\sum_0^n {}_r \sum_r^n {}_\rho \frac{1}{\rho} \binom{\rho}{r} (-1)^r u^{-r} \Delta u^r.$$

We see that $\sum_0^\rho {}_r \frac{(-1)^{\rho-r} \rho!}{r!\,(\rho-r)!} u^{-r} \Delta u^r = \Delta\left(\frac{u}{\alpha} - 1\right)^\rho$ where, when the operation Δ has been performed (α being treated as a constant), α is replaced by u. We have therefore from (7),

8. $$\Delta\phi(u) = \sum_0^n {}_\rho \frac{u^\rho \phi^\rho(u)}{\rho!} \Delta\left(\frac{u}{\alpha} - 1\right)^\rho,$$

where, after performance of operation Δ, α is replaced by u.

In particular, when u is a function of one variable (x) only, and we put $\Delta \equiv \left(\frac{d}{dx}\right)^n$ in (8), we obtain $\left(\frac{d}{dx}\right)^n \phi(u) = \sum_1^n {}_\rho \frac{u^\rho \phi^\rho(u)}{\rho!} \left(\frac{d}{dx}\right)^n \left(\frac{u}{\alpha} - 1\right)^\rho$, a symbolic expression for $\left(\frac{d}{dx}\right)^n \phi(u)$ given by Bertrand.*

Again we have $\varphi\left(u\frac{d}{du}\right) u^n = \varphi(n) u^n = E_y^n \varphi(y) u^y$, $(y = 0)$, whence $\varphi\left(u\frac{d}{du}\right) \sum a_n u^n = \sum a_n E_y^n \varphi(y) u^y$ $(y=0)$ and

9. $$\varphi\left(u\frac{d}{du}\right) \chi(u) = \chi(E_y) \varphi(y) u^y, \quad (y=0).$$

We might notice in passing that putting $u = e^x$, $\chi(u) = \psi(x)$, $\chi(E_y) = \psi\left(\frac{d}{dy}\right)$ we obtain the theorem

10. $$\varphi\left(\frac{d}{dx}\right) \psi(x) = \psi\left(\frac{d}{dy}\right) . \varphi(y) e^{xy} = \psi\left(x + \frac{d}{dy}\right) \varphi(y), \quad (y=0).$$

If in (4) we put $\phi(u) = (\log u)^s$, we get, with the help of (9),

$$\Delta (\log u)^s = \sum_0^p {}_r \frac{u^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} \frac{\theta\left(u\frac{d}{du}\right)}{u\frac{d}{du} - \alpha_r} (\log u)^s = \sum_0^p {}_r \frac{u^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} . (\log E_y)^s \frac{\theta(y) u^y}{y - \alpha_r}$$

$$= \sum_0^p {}_r \frac{u^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} . \left(\frac{d}{dy}\right)^s \frac{\theta(y) u^y}{y - \alpha_r}$$

$$= \sum_0^p {}_r \frac{u^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} u^y \left(\log u + \frac{d}{dy}\right)^s \frac{\theta(y)}{y - \alpha_r}, \quad (y=0),$$

thus,

11. $$\Delta (\log u)^s = \left(\log u + \frac{d}{dm}\right)^s \sum_0^p {}_r \frac{u^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} . \frac{\theta(m)}{m - \alpha_r}, \quad (m=0).$$

We might also obtain this result directly from (1) by differentiating with regard to m, thus

$$\Delta (\log u)^s u^m = \left(\frac{d}{dm}\right)^s \Delta u^m = \left(\frac{d}{dm}\right)^s . u^m (u^{-m} \Delta u^m) = u^m \left(\log u + \frac{d}{dm}\right)^s u^{-m} \Delta u^m;$$

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* Cal. Diff., p. 141.

substituting in this formula for Δu^m from (1), we obtain

$$\Delta (\log u)^s u^m = u^m \left(\log u + \frac{d}{dm}\right)^s \sum_0^p {}_r \frac{u^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} \, \frac{\theta(m)}{m - \alpha_r},$$

and on putting $m = 0$ in this formula we obtain (11).

Supposing $(\log u)^s$ to be developed in powers of $x, y, z \ldots .$, put in particular $u_0 = 1$ (therefore $\log u_0 = 0$),

$$\Delta \equiv \left(\frac{d}{dx}\right)^\rho \left(\frac{d}{dy}\right)^\sigma \left(\frac{d}{dz}\right)^\tau \ldots ., \quad p = n = \rho + \sigma + \tau + \ldots . ;$$

also designate by $A_{s, \rho, \sigma, \tau \ldots .}$ the coefficient of $x^\rho y^\sigma z^\tau \ldots .$ in the development of $(\log u)^s$, then (11) gives

$$A_{s, \rho, \sigma, \tau \ldots .} = \frac{1}{\rho!\, \sigma!\, \tau! \ldots .} \left(\frac{d}{dx}\right)^\rho \left(\frac{d}{dy}\right)^\sigma \left(\frac{d}{dz}\right)^\tau \ldots . (\log u)^s \Big|_{0 = x = y = z = \ldots .}$$

$$= \frac{1}{\rho!\, \sigma! \ldots .} \left(\log u_0 + \frac{d}{dm}\right)^s \sum_0^n {}_r \frac{u_0^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} \, \frac{\theta(m)}{m - \alpha_r} \Big|_{m = 0 = x = y = \ldots .}.$$

We have therefore

$$12. \qquad A_{s, \rho, \sigma, \tau \ldots .} = \frac{1}{\rho!\, \sigma! \ldots .} \left(\frac{d}{dm}\right)^s \sum_0^n {}_r \frac{\Delta u^{\alpha_r}}{\theta'(\alpha_r)} \cdot \frac{\theta(m)}{m - \alpha_r} \Big|_{m = 0 = x = y = \ldots .}$$

$$= \left(\frac{d}{dm}\right)^s \sum_1^n {}_r \frac{B_{\alpha_r, \rho, \sigma, \ldots .}}{\theta'(\alpha_r)} \cdot \frac{\theta(m)}{m - \alpha_r}$$

$$= \left(\frac{d}{dm}\right)^s \sum_1^n {}_r \frac{(-1)^{n-r} B_{r, \rho, \sigma \ldots .}}{r!\,(n-r)!} \cdot \frac{\theta(m)}{m - r}, \quad (m = 0), \quad \theta(m) = m(m-1) \ldots (m-n).$$

In the same manner that we obtained (4) from (1), from (11) we obtain

$$13. \qquad \Delta \Phi (\log u) = \Phi \left(\log u + \frac{d}{dm}\right) \sum_0^p {}_r \frac{u^{-\alpha_r} \Delta u^{\alpha_r}}{\theta'(\alpha_r)} \cdot \frac{\theta(m)}{m - \alpha_r}, \quad (m = 0).$$

Putting $u = \chi(v)$, from (4) we get

$$\Delta u^{\alpha_r} = \Delta \{\chi(v)\}^{\alpha_r} = \sum_0^p {}_s \frac{v^{-\beta_s} \Delta v^{\beta_s}}{\theta_1'(\beta_s)} \cdot \frac{\theta_1\left(v \dfrac{d}{dv}\right)}{v \dfrac{d}{dv} - \beta_s} \{\chi(v)\}^{\alpha_r}.$$

Substituting in (4),

$$\Delta \Phi \chi(v) = \sum_0^p {}_r \sum_0^p {}_s \frac{u^{-\alpha_r} \theta_2\left(u \dfrac{d}{du}\right)}{\theta_2'(\alpha_r)\left(u \dfrac{d}{du} - \alpha_r\right)} \Phi(u) \cdot \frac{v^{-\beta_s} \Delta v^{\beta_s}}{\theta_1'(\beta_s)} \cdot \frac{\theta_1\left(v \dfrac{d}{dv}\right)}{v \dfrac{d}{dv} - \beta_s} \{\chi(v)\}^{\alpha_r},$$

where

$\theta_1(m)=(m-\beta_0)(m-\beta_1)\dots(m-\beta_p)$, $\theta_2(m)=(m-\alpha_0)(m-\alpha_1)\dots(m-\alpha_p)$.

By successive applications of (4) we will get in general

$$14.\quad \Delta\Phi_\kappa\Phi_{\kappa-1}\dots\Phi_1(u)=\sum_0^p{}_q\sum_0^p{}_r\dots\sum_0^p{}_s\sum_0^p{}_t\sum_0^p{}_w\frac{\Phi_{\kappa-1}^{-\alpha_{\kappa,q}}\theta_\kappa\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}}\right)}{\theta'_\kappa(\alpha_{\kappa,q})\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}}-\alpha_{\kappa,q}\right)}\Phi_\kappa$$

$$\times\frac{\Phi_{\kappa-2}^{-\alpha_{\kappa-1,r}}\theta_{\kappa-1}\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}\right)}{\theta'_{\kappa-1}(\alpha_{\kappa-1,r})\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}-\alpha_{\kappa-1,r}\right)}\Phi_{\kappa-1}^{\alpha_{\kappa,q}}\dots\frac{\Phi_1^{-\alpha_{2,t}}\theta_2\left(\Phi_1\frac{d}{d\Phi_1}\right)}{\theta'_2(\alpha_{2,t})\left(\Phi_1\frac{d}{d\Phi_1}-\alpha_{2,t}\right)}\Phi_2^{\alpha_{3,s}}$$

$$\times\frac{u^{-\alpha_{1,w}}\theta_1\left(u\frac{d}{du}\right)}{\theta'_1(\alpha_{1,w})\left(u\frac{d}{du}-\alpha_{1,w}\right)}\Phi_1^{\alpha_{2,t}}.\,\Delta u^{\alpha_{1,w}},$$

where $\theta_\nu(m)\equiv(m-\alpha_{\nu,0})(m-\alpha_{\nu,1})\dots\dots(m-\alpha_{\nu,p})$, the α_ν's being any p *distinct* quantities taken at will, of which any number may happen to coincide with α_μ's in any other function $\theta_\mu(m)\equiv(m-\alpha_{\mu,0})(m-\alpha_{\mu,1})\dots\dots(m-\alpha_{\mu,p})$; putting for all values of ν, $\theta_\nu(m)=m(m-1)\dots\dots(m-p)\equiv\theta(m)$, we have

$$15.\quad \Delta\Phi_\kappa\Phi_{\kappa-1}\dots\dots\Phi_1(u)=\sum_0^p{}_{q,r,\dots\dots s,t,w}\frac{\Phi_{\kappa-1}^{-q}\theta\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}}\right)}{\theta'(q)\left(\Phi_{\kappa-1}\frac{d}{d\Phi_{\kappa-1}}-q\right)}\Phi_\kappa$$

$$\times\frac{\Phi_{\kappa-2}^{-r}\theta\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}\right)}{\theta'(r)\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}-r\right)}\Phi_{\kappa-1}^q\dots\dots\frac{\Phi_1^{-t}\theta\left(\Phi_1\frac{d}{d\Phi_1}\right)}{\theta'(t)\left(\Phi_1\frac{d}{d\Phi_1}-t\right)}\Phi_2^s$$

$$\times\frac{u^{-w}\theta\left(u\frac{d}{du}\right)}{\theta'(w)\left(u\frac{d}{du}-w\right)}\Phi_1^t.\,\Delta u^w.$$

For brevity I have put $\Phi_1\equiv\Phi_1(u)$, $\Phi_2\equiv\Phi_2(\Phi_1)\equiv\Phi_2\{\Phi_1(u)\}$, etc., $\Phi_\kappa\equiv\Phi_\kappa(\Phi_{\kappa-1})$. Putting in (15)

$$\frac{\theta\left(\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}\right)}{\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}-r}\Phi_{\kappa-1}^q=\frac{\Phi_{\kappa-2}^{p+1}\left(\frac{d}{d\Phi_{\kappa-2}}\right)^{p+1}}{\Phi_{\kappa-2}\frac{d}{d\Phi_{\kappa-2}}-r}\Phi_{\kappa-1}^q$$

$$=\Phi_{\kappa-2}^r\int\Phi_{\kappa-2}^{p-r}\left(\frac{d}{d\Phi_{\kappa-2}}\right)^{p+1}\Phi_{\kappa-1}^q d\Phi_{\kappa-2},$$

where, as in (6), the integration is supposed to be performed by partial integrals, the arbitrary constant being put equal to 0, and

$$\frac{1}{\theta'(r)} = \frac{(-1)^{p-r}}{r!\,(p-r)!} \equiv \frac{(-1)^{p-r}}{p!}\binom{p}{r},$$

with similar substitutions with regard to q, s, t, etc., we get for (15)

$$16.\quad \Delta\Phi_\kappa\Phi_{\kappa-1}\ldots\Phi_1(u) = \frac{(-1)^{\kappa p}}{(p!)^\kappa}\sum_0^p{}_{q,\,r,\,\ldots\,t,\,w,}\,(-1)^{q+r+\ldots+t+w}\binom{p}{q}\binom{p}{r}\ldots$$

$$\ldots\binom{p}{t}\binom{p}{w}\int\Phi_{\kappa-1}^{p-q}\left(\frac{d}{d\Phi_{\kappa-1}}\right)^{p+1}\Phi_\kappa d\Phi_{\kappa-1}\int\Phi_{\kappa-2}^{p-r}\left(\frac{d}{d\Phi_{\kappa-2}}\right)^{p+1}\Phi_{\kappa-1}^q d\Phi_{\kappa-2}\ldots$$

$$\ldots\int\Phi_1^{p-t}\left(\frac{d}{d\Phi_1}\right)^{p+1}\Phi_2^s d\Phi_1\int u^{p-w}\left(\frac{d}{du}\right)^{p+1}\Phi_1^t du\,.\,\Delta u^w.$$

For example, if we have what I might call the κ-storied function $e^{e^{\cdot^{\cdot^{\cdot^{e^u}}}}}$, where e occurs κ times, and which I will designate by u_κ, from (16) we obtain

$$17.\quad \Delta u_\kappa = \frac{(-1)^{\kappa p}}{(p!)^\kappa}\sum_0^p{}_{q,\,r,\,\ldots\,t,\,w}\,(-1)^{q+r+\ldots+t+w}(qr\ldots st)^{p+1}\binom{p}{q}\binom{p}{r}\ldots$$

$$\ldots\binom{p}{t}\binom{p}{w}\int u_{\kappa-1}^{p-q}e^{u_{\kappa-1}}du_{\kappa-1}\int u_{\kappa-2}^{p-r}e^{qu_{\kappa-2}}du_{\kappa-2}\ldots$$

$$\ldots\int u_1^{p-t}e^{su_1}du_1\int u^{p-w}e^{tu}du\,.\,\Delta u^w.$$

Where each integration is supposed to be performed by partial integrals, the arbitrary constant being put equal to 0, or in other words, that particular integral being selected which explicitly contains as a factor the exponential which appears under the integral sign in question.

We might almost indefinitely generalize the above formulae, thus, instead of u^m in (1) we might suppose Φ a function of any number of variables x, y, $z\ldots$ and of any number of parameters m_1, m_2, $m_3\ldots$, and Δ any operator independent of these parameters, and such that

$$\Delta\Phi(x, y\ldots m_1, m_2\ldots) = f(m_1, m_2, \ldots m_\kappa\ldots)\,.\,\Phi(x, y\ldots m_1, m_2\ldots),$$

where as regards the parameters alone f is of a finite degree. Supposing f to be of degrees n_1, n_2, etc. in m_1, m_2, etc. respectively, and taking any arbitrary polynomials in m's such as $\theta_\kappa(m_\kappa) = \Pi(m_\kappa - \alpha_{\kappa,t})$ where degree of polynomial is

$\nless n_\kappa$, by successive application of the theory of partial fractions as regards m_1, m_2, etc. to f, we may express it as a summation in the form

$$f(m_1,\, m_2 \,.\,.\,.\,.\, m_\kappa \,.\,.\,.\,.) = \theta_1(m_1)\,\theta_2(m_2)\,.\,.\,.\,.\sum \frac{f(\alpha_{1,r},\, \alpha_{2,s}\,.\,.\,.\,.)}{\theta_1'(\alpha_{1,r})\,\theta_2'(\alpha_{2,s})\,.\,.\,.\,.}\cdot\frac{1}{(m_1-\alpha_{1,r})(m_2-\alpha_{2,s})\,.\,.\,.\,.}$$

whence

$$\Delta\Phi(x,\, y\,.\,.\,.\,.\, m_1,\, m_2\,.\,.\,.\,.) = f(m_1,\, m_2\,.\,.\,.\,.)\,\Phi(x,\, y\,.\,.\,.\,.\, m_1,\, m_2\,.\,.\,.\,.)$$

$$= \Pi\theta_\kappa(m_\kappa)\sum_{r,s\,.\,.\,.\,.}\frac{f(\alpha_{1,r},\, \alpha_{2,s}\,.\,.\,.)\,\Phi(x,\, y\,.\,.\,.\,.\,\alpha_{1,r},\, \alpha_{2,s},\,.\,.\,.\,.)}{\theta_1'(\alpha_{1,r})\,\theta_2'(\alpha_{2,s})\,.\,.\,.\,.} \times \frac{1}{(m_1-\alpha_{1,r})(m_2-\alpha_{2,s})\,.\,.\,.\,.}\cdot\frac{\Phi(x,\, y\,.\,.\,.\,.\, m_1,\, m_2\,.\,.\,.\,.)}{\Phi(x,\, y\,.\,.\,.\,.\,\alpha_{1,r},\, \alpha_{2,s}\,.\,.\,.\,.)}$$

$$= \Pi\theta_\kappa(m_\kappa)\sum_{r,s\,.\,.\,.\,.}\frac{\Delta\Phi(x,\, y\,.\,.\,.\,.\,\alpha_{1,r},\, \alpha_{2,s}\,.\,.\,.\,.)}{\theta_1'(\alpha_{1,r})\,\theta_2'(\alpha_{2,s})\,.\,.\,.\,.} \times \frac{1}{(m_1-\alpha_{1,r})(m_2-\alpha_{2,s})\,.\,.\,.\,.}\cdot\frac{\Phi(x,\, y\,.\,.\,.\,.\, m_1,\, m_2\,.\,.\,.\,.)}{\Phi(x,\, y\,.\,.\,.\,.\,\alpha_{1,r},\, \alpha_{2,s}\,.\,.\,.\,.)}.$$

Bernoulli's Numbers.

We might notice the application of (7) to Bernoulli's Numbers; in (7) put $\Delta \equiv \left(\frac{d}{dx}\right)^n$, $u = e^x$, $\Phi(u) = (1+u)^{-1}$, then $u_0 = 1$ and

$$\left(\frac{d}{dx}\right)^n (1+e^x)^{-1}_{x=0}$$

$$= \sum_{0}^{n}{}_r \sum_{r}^{n}{}_\rho \frac{(-1)^{\rho-r} u_0^{\rho-r}\Phi^\rho(u_0)}{r!\,(n-r)!}\left(\frac{d}{dx}\right)^n_{x=0} e^{rx} = \sum_{1}^{n}{}_r \sum_{r}^{n}{}_\rho \frac{(-1)^r \rho!}{r!\,(\rho-r)!}\cdot\frac{r^n}{2^{\rho+1}}$$

$$= \sum_{1}^{n}{}_r \sum_{r}^{n}{}_\rho \frac{(-1)^r r^n}{r!\;2^{r+1}}\left(\frac{d}{dx}\right)^r_{x=\frac{1}{2}} x^\rho = \sum_{1}^{n}{}_r \frac{(-1)^r r^n}{r!\;2^{r+1}}\left(\frac{d}{dx}\right)^r_{x=\frac{1}{2}} \frac{1-x^{n+1}}{1-x}$$

$$= \sum_{1}^{n}{}_r \frac{(-1)^r r^n}{r!\;2^{r+1}}\left\{\left(\frac{d}{dx}\right)^r \frac{1}{1-x} - \sum_{0}^{r}{}_s \frac{r!}{s!\,(r-s)!}\left(\frac{d}{dx}\right)^s x^{n+1}.\left(\frac{d}{dx}\right)^{r-s}\frac{1}{1-x}\right\}_{x=\frac{1}{2}}$$

$$= \sum_{1}^{n}{}_r (-1)^r r^n\left\{1-\frac{1}{2^{n+1}}\sum_{0}^{r}{}_s \binom{n+1}{s}\right\} = \frac{1}{2^{n+1}}\sum_{1}^{n}{}_r \sum_{r+1}^{n+1}{}_s (-1)^r r^n \binom{n+1}{s}$$

$$= \frac{1}{2^{n+1}}\sum_{2}^{n+1}{}_s \sum_{1}^{s-1}{}_r \binom{n+1}{s}(-1)^r r^n = \frac{1}{2^{n+1}}\sum_{1}^{n}{}_r (-1)^r r^n M_{n-r} = \frac{1}{2^{n+1}}\sum_{2}^{n+1}{}_s \binom{n+1}{s} N_{s-1}$$

where $$M_t \equiv 1 + \binom{n+1}{1} + \binom{n+1}{2} + \ldots + \binom{n+1}{t}$$

and $$N_t = -1^n + 2^n - 3^n + \ldots + (-1)^t t^n.$$

We know that Bernoulli's p^{th} number $B_p = \frac{(-1)^p(2p)!}{2^{2p}-1}$ multiplied by coefficient of x^{2p-1} in $(1+e^x)^{-1}$, putting $n \equiv 2p-1$, we have therefore

18. $$Bp = \frac{2p(-1)^p}{2^{2p}-1}\left(\frac{d}{dx}\right)^{2p-1}(1+e^x)^{-1}_{x=0}$$

$$= \frac{(-1)^p p}{2^{2p-1}(2^{2p}-1)}\sum_{1}^{2p-1}{}_r(-1)^r r^{2p-1} M_{2p-r-1} = \frac{(-1)^p p}{2^{2p-1}(2^{2p}-1)}\sum_{2}^{2p}{}_s\binom{2p}{s} N_{s-1}.$$